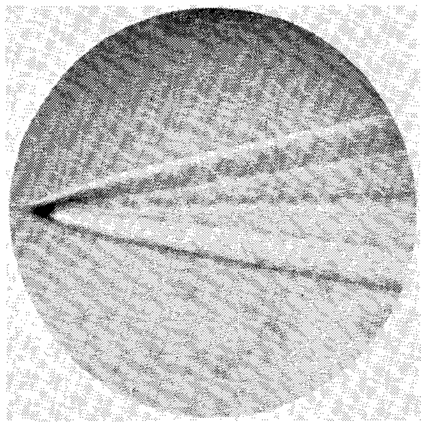

Fig. 1 25° included angle cone in normal flight (5500 fps, 25 mm Hg air pressure)

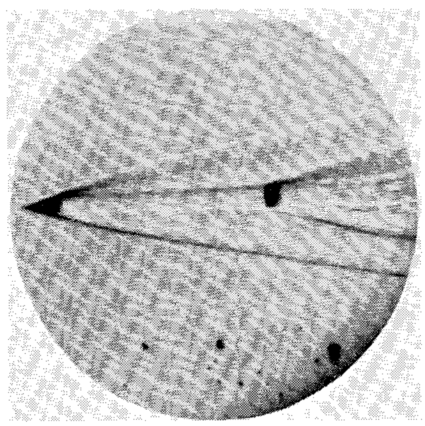

Fig. 2 25° included angle cone followed by a piece of sabot (5500 fps, 25 mm Hg air pressure)

schlieren photography, and assurance of constant attitude have been described elsewhere.[1] Note that the characteristics of the flow are as expected. There are bow and secondary shocks, and the laminar trail is much narrower than the base of the cone. There is a distinct necking of the flow, causing the secondary shock, and, further downstream, there is a distinct laminar-to-turbulent transition. Hundreds of pictures like this have been taken under various flight conditions.

Figure 2 shows an identical cone fired under precisely the same conditions, but with the base of the sabot located in the wake of the cone and some distance behind it. Note that the flow behind the cone is completely different in this case. There is no secondary shock. There is no necking of the flow. The trail is still laminar but of approximately the diameter of the base. No hint of laminar to turbulent transition exists ahead of the sabot. Examination of the film density of the schlieren photograph indicates that, as in Fig. 1, the laminar trail is a region of much lower gas density than the surrounding inviscid region (an expected temperature and pressure phenomenon in the case of the normal flow this close to the body). Obviously, the cone and sabot have interacted via the wake of the cone. This is *not* an isolated phenomenon; the results are completely reproducible.

The first and obvious lesson from these two photographs is that considerable care must be taken by researchers in ballistic ranges to insure that sabots separate properly and that the field of view of their optics is sufficiently wide so that the possibility of the interaction between two or more projectiles is eliminated. Otherwise, the flow characteristics photographed are not necessarily those of the body under study.

[1] Slattery, R. E. and Clay, W. G., "The turbulent wake of hypersonic bodies," ARS Preprint 2673-62 (1962).

Figure 2 is interesting in and for itself. The sabot, traveling in the wake of the cone, has made its presence felt upstream and has changed the otherwise normal flow about the cone. The simplest interpretation is that the sabot is immersed in a fluid with respect to which it has a *subsonic* velocity, despite its high velocity in the laboratory system. Under these conditions it can propagate energy back up the cone's trail, countercurrent to the flow in the trail (in the body-centered system), and alter the characteristic flow about the cone. This comes about, probably, for two reasons: 1) the flow in the wake of the cone is quite high speed in the observer system and is an appreciable fraction of the velocity of the sabot; and 2) the flow is hot, which tends to raise the sound speed.

Of course, having stated that energy is propagated up the trail by no means describes the details of the processes that alter the normal flow.

Invariant Components of Motion in Inverse-Square Force Fields

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CRONIN and Schwartz[1] have drawn attention to a useful, but little known, property of motion in a two-body motion in an inverse-square force field, namely, that the velocity vector can be resolved at any point into two components of constant magnitude, one remaining normal to the initial line and the other remaining normal to the radius vector. This property of the motion also has been proved in the well-known text on dynamics by Whittaker.[2] The present note is a brief outline of work[3] published in 1959 which used the same invariant properties and applied the method to the problem of small drag and low thrust.

Kepler's second law states that the radius vector sweeps out equal areas in equal times; the quantity $r^2(d\vartheta/dt) = h$ is a constant of the motion and is, of course, a first integral of the equations of motion. Here ϑ is the true anomaly and h is a constant.

The existence of an invariant such as $r^2(d\vartheta/dt)$ immediately suggests the problem of finding additional invariants, and this search is successful if the invariant components just noted are used. If v_1 denotes the component normal to the initial line and v_2 denotes the component normal to the radius vector (V_p and V_h, respectively, in Fig. 2 of Ref. 1), then it also is of interest to note that v_1/v_2 is the eccentricity of the orbit. The square of the speed then is given by

$$v^2 = v_1{}^2 + v_2{}^2 + 2v_1v_2 \cos(\vartheta)$$

If, further, a is the semimajor axis of the elliptical orbit, e the eccentricity of the orbit, and R the radius of the earth, the three invariants of the motion can be written as

$$\begin{aligned} x &= p(d\vartheta/dt) - (dp/dt)\cot(\vartheta) = (R/L)^{1/2} \\ y &= (dp/dt)/\sin(\vartheta) = e(R/L)^{1/2} \\ z &= p^2(d\vartheta/dt) = (L/R)^{1/2} \end{aligned} \qquad (1)$$

In Eqs. (1), $p = r/R$ and $L = a(1 - e^2)$; for convenience, the area integral is denoted by z, and v_1 and v_2 now are denoted by y and x, respectively. The quantities x and z are dependent in the classical case; in the following, x,y, and z will be used as new dependent variables, with ϑ as the new independent variable. The original dependent variables were r and ϑ as functions of the time t.

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In the motion of a satellite near the earth, an important force, in addition to the gravitational force, is that due to atmospheric drag. If it is assumed that the drag force is tangential to the path and proportional to the density and to the square of the speed, then the basic equations in the original variables can be written as

$$\begin{aligned} d^2r/dt^2 - r(d\vartheta/dt)^2 + K/r^2 &= (-C_DA/2m)\rho(dr/dt)[(dr/dt)^2 + r^2(d\vartheta/dt)^2]^{1/2} \\ (1/r)(d/dt)(r^2d\vartheta/dt) &= (-C_DA/2m)\rho r(d\vartheta/dt)[(dr/dt)^2 + r^2(d\vartheta/dt)^2]^{1/2} \end{aligned} \quad (2)$$

where ρ is the density, m the mass of the satellite, A the normal cross-sectional area, and C_D the drag coefficient assumed as constant.

If the satellite is sufficiently high, then the drag terms may be dropped, and the analysis leads to the usual Kepler results; if the satellite is near the re-entry condition, then the drag forces will dominate the gravitational forces, and simplifying assumptions can be made.[4] However, in the studies of the lifetime of a satellite, or in orbital studies where it is assumed that the satellite is several revolutions away from the re-entry condition, the accurate inclusion of the drag terms becomes necessary.

Roberson[5] analyzed Eqs. (2) by a formal perturbation procedure after the introduction of new variables. Since the quantity z in Eq. (1) is constant in the drag-free case, it is to be expected that this quantity would vary slowly as a function of the time in the presence of drag forces. Roberson used R/r and $KR/(r^2d\vartheta/dt)^2$ as new variables and was able to reduce the original equations to a second-order equation that was linear and a first-order equation. However, Roberson used only one dynamical invariant in the analysis. It has been noted that there are two components of velocity which are invariant in the two-body motion. Without any essential restriction in generality, the dynamical equations can be written for the case of constant tangential thrust; the modifications in the case of drag are clear.

It is of interest to change to dimensionless variables; the dimensionless time $\tau = (R/g)^{1/2}t$, where g is the acceleration due to gravity at the surface of a spherical earth, and the dimensionless velocity is V, where $v = (gR)^{1/2}V$. The nondimensional measure of the thrust may be denoted by μ. The dynamical equations in (2) may be written as

$$\begin{aligned} d^2p/d\tau^2 - p(d\vartheta/d\tau)^2 + (1/p^2) &= \mu(dp/d\tau)/V \\ p(d^2\vartheta/d\tau^2) + 2(dp/d\tau)(d\vartheta/d\tau) &= \mu p(d\vartheta/d\tau)/V \end{aligned} \quad (3)$$

If the definitions used in Eqs. (1) are used in Eqs. (3), one may write

$$\begin{aligned} p &= z/(x + y \cos\varphi) \\ dp/d\tau &= y \sin\vartheta \\ d\vartheta/d\tau &= (x + y \cos\vartheta)^2/z \end{aligned} \quad (4)$$

The first relation in Eqs. (4) shows that, for constant (x,y,z), p is in the correct polar form for the equation of the undisturbed orbit. In the Kepler case these constants are known, and the orbit is fixed. In the case of small thrust or drag, (x,y,z) should be slowly varying functions of ϑ and therefore of the time t. There are several advantages to be gained from working with the orbital equations in the form of Eqs. (4); the first equation, for example, gives the instantaneous ellipse at any instant of time if (x,y,z) are known at that time. Also (y/x) is the instantaneous eccentricity of the orbit. The functions (x,y,z) can be determined successively from the solution of first-order equations; in each case the starting approximations are known constants.

To indicate the work briefly, the first and second equations of (4) must be related; if primes denote differentiations with respect to the angle ϑ, this relation can be written as

$$x'z + y'z \cos(\vartheta) - z'(x + y \cos\vartheta) = 0 \quad (5)$$

The second relation is determined conveniently by dividing the left and right sides, respectively, of Eqs. (3). This equation ultimately[3] can be written as

$$y'(-z \sin\vartheta) + z'(-y \sin\vartheta) = 1 - xz \quad (6)$$

Equation (5), of course, does not depend upon dynamical considerations; in the Kepler case $xz = 1$, and the right side of Eq. (6) vanishes for the starting approximation. This equation does not depend upon the force law, but the force is required to remain tangential to the orbit. This equation does not involve the density, magnitude of the velocity, or any thrust parameter. The third and final equation does involve the thrust parameter:

$$z' = (\mu z^2/V)/(x + y \cos\vartheta)^2 \quad (7)$$

The three first-order equations in (x,y,z) as functions of ϑ are Eqs. (5–7).

It now is assumed that the solutions may be written in the form of the perturbation series solution:

$$\begin{aligned} x(\vartheta) &\quad x_0 + \mu x_1(\vartheta) + \mu^2 x_2(\vartheta) + \ldots \\ y(\vartheta) &\quad y_0 + \mu y_1(\vartheta) + \mu^2 y_2(\vartheta) + \ldots \\ z(\vartheta) &\quad z_0 + \mu z_1(\vartheta) + \mu^2 z_2(\vartheta) + \ldots \end{aligned} \quad (8)$$

and it is known that, if $\mu = 0$, $x,y,$ and z are constants [Eqs. (1)]. The terms with subscripts unity then can be written down in terms of integrals of known functions.

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Comments

Errata

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AND

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THE authors would like to call attention to the following misprints that appeared in the paper "On Calculations of the Laminar Separation Point, and Results for Certain Flows," by Morris Morduchow and Stanley P. Reyle, in the Readers' Forum of the *Journal of the Aerospace Sciences*, August 1962, p. 996.

In Eq. (2), the exponent should read "1/(6.13n-1)." In Eq. (3), the exponent should read "1/(6.13n + 1)." In the fourth line after Eq. (3), the beginning of the sentence should read "For $u_1/u_\infty = 1 - \xi^n (n > 0)$ . . . ."

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